State Space models and the Kalman filter

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Uses of state space models and the Kalman filter

Can be used to answer questions such as:

• Has the relationship between Y and X changed over time
• Does GDP have a permanent and transitory component?
• Is there a common factor driving energy prices across the world?
What is a state space model?

A state space model consists of two equations:

**Measurement equation:** An equation that describes the relationship between observed and unobserved variables.

**Transition equation:** An equation that describes the dynamics of the unobserved variables.
What is a state space model?

The two equations are as follows:

**Observation eq**

\[ y_t = H_t B_t + A z_t + e_t \]

**Transition eq**

\[ \beta_t = \mu + F \beta_{t-1} + v_t \]
What is a state space model?

The shocks

\[ y_t = H_t B_t + A z_t + e_t \]
\[ \beta_t = \mu + F \beta_{t-1} + v_t \]

Assumptions

\[ e_t \sim iid. N(0, R) \]
\[ v_t \sim iid. N(0, Q) \]
\[ E(e_t, v_t) = 0 \]
Examples of econometric models in state space form

• A time-varying parameter model linking import price inflation and the nominal exchange rate

\[ \Delta p_t = \alpha_t + D_t E_t + e_t \]
A time-varying parameter model

• In SS form

\[ \Delta p_t = \alpha_t + D_tE_t + e_t \]

Unobserved state variables

Matrix \( H \)

Transition equation with \( \mu = 0, F = 1 \)
Examples: A trend-Cycle model

- Decomposing GDP into a permanent (Trend) and transitory (Cycle) component

\[ GDP_t = T_t + C_t \]

- We assume that the trend component follows a random walk

\[ T_t = c + T_{t-1} + v_{1,t} \]

- The cyclical component is a stationary process

\[ C_t = \rho C_{t-1} + v_{2,t} \]
Trend-Cycle model

- In SS form

\[ GDP_t = [1 \ 1] \begin{pmatrix} T_t \\ C_t \end{pmatrix} \]

\[ \begin{pmatrix} T_t \\ C_t \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ \rho & 0 \end{pmatrix} \begin{pmatrix} T_{t-1} \\ C_{t-1} \end{pmatrix} + \begin{pmatrix} v_{1,t} \\ v_{2,t} \end{pmatrix} \]
Trend-Cycle model

SS form with AR(2) cycle

\[
GDP_t = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{pmatrix} T_t \\ C_t \\ C_{t-1} \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_{t-1} \\ C_{t-1} \\ C_{t-2} \end{pmatrix} + \begin{pmatrix} v_{1t} \\ v_{2t} \\ 0 \end{pmatrix}
\]
Examples: A dynamic factor model

- Common factor in GDP growth across Latin America

\[ \Delta \text{gdp}_{i,t} = B_i F_t + e_{it} \]

GDP growth in country i, i=1..4

- Assume factor follows AR process

\[ F_t = c + \rho_1 F_{t-1} + \rho_2 F_{t-2} + \nu_t \]
A dynamic factor model

• SS form

\[
\begin{pmatrix}
\Delta gdp_{1,t} \\
\Delta gdp_{2,t} \\
\Delta gdp_{3,t} \\
\Delta gdp_{4,t}
\end{pmatrix}
= \begin{pmatrix}
B_1 & 0 \\
B_2 & 0 \\
B_3 & 0 \\
B_4 & 0
\end{pmatrix}
\begin{pmatrix}
F_t \\
F_{t-1}
\end{pmatrix}
+ \begin{pmatrix}
e_{1t} \\
e_{2t} \\
e_{3t} \\
e_{4t}
\end{pmatrix}
\]

\[
\begin{pmatrix}
F_t \\
F_{t-1}
\end{pmatrix}
= \begin{pmatrix}
\rho_1 & \rho_2 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
F_{t-1} \\
F_{t-2}
\end{pmatrix}
+ \begin{pmatrix}
v_{1t} \\
v_{2t}
\end{pmatrix}
\]
Estimation Aims

Aim 1: Estimate parameters of the State Space

\[ y_t = H_t B_t + A z_t + e_t \]
\[ \beta_t = \mu + F \beta_{t-1} + \nu_t \]
\[ e_t \sim iid. N(0, R) \]
\[ \nu_t \sim iid. N(0, Q) \]
Aim 2: Recover the unobserved state

\[ y_t = H_t B_t + A z_t + e_t \]

\[ \beta_t = \mu + F \beta_{t-1} + \nu_t \]

\[ e_t \sim \text{iid. } N(0, R) \]

\[ \nu_t \sim \text{iid. } N(0, Q) \]
Estimation

• We use the **Kalman filter** for this purpose

• The Kalman filter is a recursive algorithm that provides an optimal estimate of $\beta_t$ conditional on an information set and knowledge of the parameters of the state space $H_t, A, \mu, F, R, Q$
Notation

Model

\[ y_t = x_t \beta_t + e_t \]
\[ \beta_t = \mu + F \beta_{t-1} + v_t \]
\[ e_t \sim \text{iid.} N(0, R) \]
\[ v_t \sim \text{iid.} N(0, Q) \]

Estimate of \( \beta_t \) conditional on information up to time \( t-1 \)

\[ \beta_{t/t-1} \]
Notation

- Estimate of $\beta_t$ conditional on information up to time $t$

$$\beta_{t/t}$$

- Covariance of $\beta_t$ conditional on information up to $t-1$

$$P_{t/t-1}$$
Notation

- Covariance of $\beta_t$ conditional on information up to $t-1$

$$P_{t/t}$$

- Forecast of $y$ given information up to time $t-1$

$$Y_{t/t-1}$$
Notation

• Prediction error

$$\eta_{t/t-1} = y_t - Y_{t/t-1}$$

• Variance of the prediction error

$$f_{t/t-1}$$
Kalman filter

- Assume parameters $\mu, F, R, Q$ are known

The Kalman filter recursion consists of three steps

1. Start with guess of state at time 0 $\beta_{0/0}$ and $P_{0/0}$

2. **Prediction**: At time 1 form an optimal prediction $y_{1/0}$ using an estimated value for $\beta_{1/0}$
3. **Updating**: Use the observed value of $y$ at time 1 to calculate the prediction error

$$
\eta_{1/0} = y_1 - y_{1/0}
$$

→ this prediction error contains information we can use to refine our guess about $\beta$

$$
\hat{\beta}_{1/1} = \beta_{1/0} + K_t \eta_{1/0}
$$

**Conditional on time 1**

weight assigned to new Information (Kalman gain)
Kalman filter equations

Prediction

\[ \beta_{t/t-1} = \mu + F\beta_{t-1/t-1} \]
\[ P_{t/t-1} = FP_{t-1/t-1}F' + Q \]
\[ \eta_{t/t-1} = y_t - y_{t/t-1} = y_t - x_t\beta_{t/t-1} \]
\[ f_{t/t-1} = x_tP_{t/t-1}x'_t + R \]
Where do the prediction equations come from?

First equation \( \beta_{t/t-1} = \mu + F\beta_{t-1/t-1} \)

Examine transition eq \( E(\mu + F\beta_{t-1} + v_t) \)

Second equation \( P_{t/t-1} = FP_{t-1/t-1}F' + Q \)

Variance of transition eq

\[ \text{VAR}(\mu + F\beta_{t-1} + v_t) = \text{VAR}(\mu) + F \times \text{VAR}(\beta_{t-1}) \times F' + \text{VAR}(v_t) \]

\[ = 0 = P_{t-1/t-1} = Q \]
Where do these prediction equations come from?

Third equation: \( \eta_{t/t-1} = y_t - y_{t/t-1} = y_t - x_t \beta_{t/t-1} \)

Fourth equation: \( f_{t/t-1} = x_t P_{t/t-1} x_t' + R \)

\[
\begin{align*}
f_{t/t-1} &= VAR(y_t - y_{t/t-1}) = VAR([x_t \beta_t + e_t] - x_t \beta_{t/t-1}) \\
&= E\left( (x_t (\beta_t - \beta_{t/t-1}) + e_t)^2 \right) \\
&= E(x_t \beta_t - \beta_{t/t-1})^2 + E(2e_t x_t (\beta_t - \beta_{t/t-1})) + E(e_t^2) \\
&= x_t P_{t/t-1} x_t' + 0 + R
\end{align*}
\]
Kalman Filter Equations

Updating

\[ \beta_{t/t} = \beta_{t/t-1} + K_t \eta_{t/t-1} \]
\[ P_{t/t} = P_{t/t-1} - K_t x_t P_{t/t-1} \]
\[ K_t = P_{t/t-1} x_t f_{t/t-1}^{-1} \]

Weight assigned to new information
In the prediction error
Kalman gain

Update estimates using information contained in the prediction error
Where do the updating equations come from?

- We need to update the forecast of $\beta_t$ based on new information contained in $\eta_t$.
- Updating a linear projection: In general if we want to calculate an updated forecast

$$F(Y_3/Y_2, Y_1) = F(Y_3/Y_1) + H_{32} H_{22}^{-1} [Y_2 - F(Y_2/Y_1)]$$
Where do these updating equations come from?

\( K_t = P_{t/t-1} x_t' f_{t/t-1} \)

\[
Y_3 = \beta_t, Y_2 = y_t, Y_1 = x_t \rightarrow \\
\beta_{t/t} = \beta_{t/t-1} + [\text{cov}(\beta_t, y_t) \text{var}(y_t)][y_t - y_{t/t-1}]
\]

Updating the variance of the forecast error \( P_{t/t} \)

Use the formula

\[
\text{VAR}(Y_3 - F(Y_3/Y_2, Y_1)) = H_{33} - H_{32} H_{22}^{-1} H_{23}
\]
The Kalman Gain

• The Kalman gain or weight given to new information (about the state) contained in the prediction error is given by

\[ K_t = P_{t/t-1} x'_t f_{t/t-1} \]

\[ = P_{t/t-1} x'_t x_t (P_{t/t-1} x'_t + R)^{-1} \]

Positive function of uncertainty associated with \( \beta_{t/t-1} \) higher weight to prediction error if this is higher

Lower weight if R is high
Overview of the Kalman filter

Starting values (time 0)
\[ \beta_{0/0}, P_{0/0} \]
\[ \downarrow \]

Predict state vector (time 1, \ldots )
\[ \beta_{t/t-1} = \mu + F \beta_{t-1/t-1} \]
\[ P_{t/t-1} = FP_{t-1/t-1}F' + Q \]
\[ \downarrow \]

Calculate prediction error
\[ \eta_{t/t-1} = y_t - y_{t/t-1} = y_t - x_t \beta_{t/t-1} \]
\[ f_{t/t-1} = x_t P_{t/t-1} x_t' + R \]
\[ \downarrow \]

Update states
\[ K_t = P_{t/t-1} x_t' f_{t/t-1}^{-1} \]
\[ \beta_{t/t} = \beta_{t/t-1} + K_t \eta_{t/t-1} \]
\[ P_{t/t} = P_{t/t-1} - K_t x_t P_{t/t-1} \]
Kalman Smoother

- The Kalman filter provides inference for the state vector using information up to time $t$ $\beta_{t/t}$.
- We can get estimate of the state vector using information up to the end of the sample $\beta_{t/T}$ using the **Kalman smoother**

$$
\beta_{t/T} = \beta_{t/t} + P_{t/t}F'P_{t+1/t}^{-1}(\beta_{t+1/T} - F\beta_{t/t} - \mu)
$$

$$
P_{t/T} = P_{t/t} + P_{t/t}F'P_{t+1/t}^{-1}(P_{t+1/T} - P_{t+1/t})(P_{t/t}F'P_{t+1/t})'
$$
Kalman Smoother

\[ \beta_{t/T} = \beta_{t/t} + P_{t/t}F'P_{t+1/t}^{-1}(\beta_{t+1/T} - F\beta_{t/t} - \mu) \]

Starting point is output of Kalman Filter at time \( T \)

Update state estimate as a weighted average of filtered estimate and the error in ‘predicting’ the state variable
Example of filtering and smoothing persistence of CPI inflation in the UK

\[ \pi_t = c_t + \beta_t \pi_{t-1} + \epsilon_t \]

\[
\begin{pmatrix}
    c_t \\
    \beta_t
\end{pmatrix}
= \begin{pmatrix}
    c_{t-1} \\
    \beta_{t-1}
\end{pmatrix} + \begin{pmatrix}
    \nu_{1t} \\
    \nu_{2t}
\end{pmatrix}
\]

\[
\begin{pmatrix}
    \epsilon_t \\
    \nu_{1t} \\
    \nu_{2t}
\end{pmatrix} \sim N \left( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.6 & 0 & 0 \\ 0 & 0.006 & 0 \\ 0 & 0 & 0.001 \end{pmatrix} \right)
\]
Filtered and Smoothed estimates

Filtered estimate of Autoregressive Coefficient

Smoothed estimate of autoregressive coefficient

Filtered estimate of constant

Smoothed estimate of constant
Kalman Filter and Maximum Likelihood estimation

- Up to now we have assumed that the parameters of the state space $H_t, A, \mu, F, R, Q$ are known
- Generally this is not the case and these parameters have to be estimated
- The Kalman filter also provides us with the likelihood function which can be maximised wrt these parameters
Maximum Likelihood

- Recall our Time-varying parameter model

\[ y_t = x_t \beta_t + e_t \]
\[ \beta_t = \mu + F \beta_{t-1} + \nu_t \]
\[ e_t \sim iid. N(0, R) \]
\[ \nu_t \sim iid. N(0, Q) \]

Given normal error terms and state vector, the data is distributed as

\[ y_t/x_t \sim N(x_t \beta_{t/t-1}, x_t P_{t/t-1} x_t' + R) \]
Maximum Likelihood

The Likelihood function is given by

\[ \ln L = -\frac{1}{2} \sum_{t=1}^{T} \ln(2\pi^n \left[ \det(x_t P_{t/t-1} x_t') + R \right]) - \frac{1}{2} \sum_{t=1}^{T} (y_t - x_t \beta_{t/t-1})' (x_t P_{t/t-1} x_t' + R)^{-1} (y_t - x_t \beta_{t/t-1}) \]

or

\[ \ln L = -\frac{1}{2} \sum_{t=1}^{T} \ln(2\pi^n \det(f_{t/t-1})) - \frac{1}{2} \sum_{t=1}^{T} \eta_{t/t-1}' f_{t/t-1}^{-1} \eta_{t/t-1} \]
Overview of the Kalman Filter with ML

Starting values (time 0)
Initial conditions for filter and guess for SS parameters

\[
\begin{align*}
\beta_{0/0}, P_{0/0}, Q_0, \mu_0, F_0, R_0 \\
\downarrow \\
\beta_{t/t-1} &= \mu + F \beta_{t-1/t-1} \\
P_{t/t-1} &= FP_{t-1/t-1}F' + Q \\
\downarrow \\
\eta_{t/t-1} &= y_t - y_{t/t-1} = y_t - x_t \beta_{t/t-1} \\
f_{t/t-1} &= x_t P_{t/t-1} x_t' + R \\
\downarrow \\
K_t &= P_{t/t-1} x_t' f_{t/t-1}^{-1} \\
\beta_{t/t} &= \beta_{t/t-1} + K_t \eta_{t/t-1} \\
P_{t/t} &= P_{t/t-1} - K_t x_t P_{t/t-1} \\
\downarrow \\
\ln L &= \ln L + \left[ -\frac{1}{2} \ln(2\pi^n \det(f_{t/t-1})) - \frac{1}{2} \eta_{t/t-1} f_{t/t-1}^{-1} \eta_{t/t-1} \right] \\
\downarrow \\
\ln L &= -\frac{1}{2} \sum_{t=1}^{T} \ln(2\pi^n \det(f_{t/t-1})) - \frac{1}{2} \sum_{t=1}^{T} \eta_{t/t-1} f_{t/t-1}^{-1} \eta_{t/t-1} 
\end{align*}
\]
State Space models in Eviews

Go to object ➔ new object ➔ sspace
Entering your state space model

An unobserved component model

\[\text{@signal inf} = sv1 + sv2\]

\[\text{@state sv1} = sv1(-1) + [\text{var} = \exp(c(2))]\]

\[\text{@state sv2} = c(4)*sv2(-1) + [\text{var} = \exp(c(3))]\]
Entering your state space model

• A time-varying parameter model

@signal inf = sv2 + sv1*inf1 + [var = exp(c(1))]

@state sv1 = sv1(-1) + [var = exp(c(2))]

@state sv2 = sv2(-1) + [var = exp(c(3))]
Specifying starting values for the parameters

\[
\text{param c(1) -2 c(2) -2 c(3) -2}
\]

\[
\text{@signal inf} = sv2 + sv1 \times \text{inf1} + [\text{var = exp(c(1))}]
\]

\[
\text{@state sv1} = sv1(-1) + [\text{var = exp(c(2))}]
\]

\[
\text{@state sv2} = sv2(-1) + [\text{var = exp(c(3))}]
\]